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DIRECTIONAL SPLINES FOR ECONOMIC ANALYTICS

Abstract. *Paper presents the method of constructing a cubic spline for a set of points on a plane. It has been made comparison of the spline with Schoenberg B-spline and Akima and Kathmul-Rom splines. It is shown that for unequally spaced points, at which the disadvantages of the named splines are usually manifested, in comparison with the B-spline, it gives significantly lower oscillations. The spline with such a set of points is practically deprived of the strong kinks that are characteristic of Akima splines. It does not give loops and oscillations, which are a characteristic disadvantage of parametric splines, in particular, Hermitian ones, which includes the Kathmull-Rom spline. The optimization method of spline guide coefficient is proposed, the purpose of which is to minimize discontinuities of the second derivative function at its intermediate points. A fourth-order spline is also proposed, which is deprived of kinks and has lower emissions compared to the Schoenberg spline. The proposed method for blunting sharp peak curves can be applied to all types of splines.*

***Keywords**: B-spline, Akima spline, Catmull-Rom spline, directional spline, spline optimization.*

JEL Classification: E00, E10, E17, E22, E27

1. Introduction

In the field of economic analytics, after data collection, they are processed. In most cases, such data is the relationship between input and output quantities as a set of points

$$(x_0,y_0),\ (x_1,y_1),\ \ldots,\ (x_{n-1},y_{n-1}),\ (x_n,y_n),\ x_{i-1}<x_i,\ i = 1,2,\ldots,n. \qquad (1)$$

In analytical reports and calculations based on it, it is often necessary to obtain a smooth curve that must pass through these points.

This problem can be solved by using interpolation methods [(Powell, 1981; Atkinson, 1988; Volkov, 2004; Watson, 1980)]. The quality of the obtained curve on the graph can be accurately estimated visually or more strictly using a mathematical formulae.

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Experience shows that for sets with a small number of points satisfactory results are obtained by the Lagrange, Newton, Stirling interpolation methods [Atkinson, 1988; Watson, 1980; Schatzman, 2002]. However, with an increase in the number of points in set (1), interpolation polynomials in the region of extreme points give unacceptably large amplitude oscillations [Schatzman, 2002].

Splines are deprived of this drawback [Schoenberg, 1946; Ahlberg et al., 1967; David et al., 1989; Cohen, 1969; Warnock, 1969; Watkins, 1970; Yanenko et al., 1970; Constantini et al., 1984; Ryabenky, 1974; Dietzeand Schmidt, 1988; Zavyalov et al., 1980; Miroshnichenko, 1995; Segeth, 2018]. The most famous among them is Schoenberg's B-spline [Schoenberg, 1946;Dobson, 1983; Hughes et al.,2005]. It ensures perfect smoothness of the curve for equidistant points x_i (i = 0,1, …, n) for which

$$h_i = x_i - x_{i-1} = h = \text{const}. \qquad (2)$$

If (2) is failed, then the smoothness of the curves, as a rule, is violated. In such cases, the *B*-spline can give significant oscillation curve (the so-called «ejections») that occur in the region of segments with small hi.

The Akima spline [Akima, 1970; Krukovetsand Gorelkin, 2019] and Hermitian splines, a special case of which is the Catmull-Rom parametric spline [Catmulland Rom, 1974; Barry and Goldman, 1988] used in graphic geometry, can be used to avoid «ejections» of the *B*-spline. However, these splines have their drawbacks.

So, the Akima spline often gives unacceptable kinks of the function graph at the nodal points, which are clearly visible in Figure 1 (curve 2).

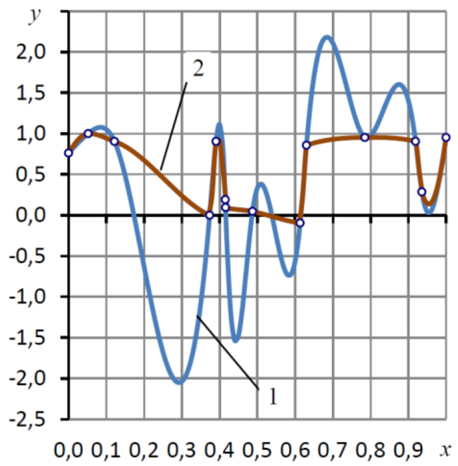

Figure 1. Graphs of the B-spline (1) and Akima spline (2)

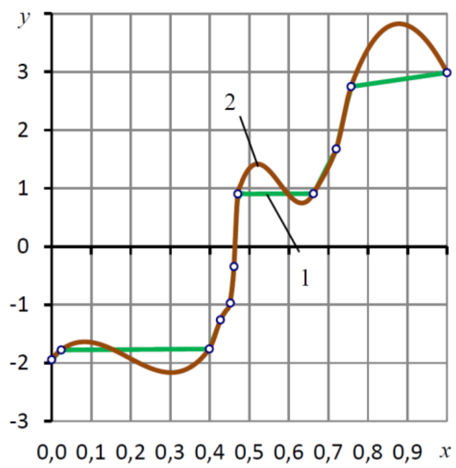

Figure 2. Linear interpolation spline (1) and Akima spline (2)

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By the way, this spline does not always justify its purpose to build monotonic functions from ordered sequences of ordinates of a data set.

Figure 2 shows an example of a graph where the mentioned sequence is ordered, but the Akima spline does not ensure the monotony of the curve constructed with its help.

Smoother transitions between neighboring polynomials are given by the Catmull-Rom splines (Figure 3). However, for small hi, they can form loops at nodal points. In addition to it, local extrema may appear on the parametric dependence $x = x(t)$, which obviously should be monotonically increasing, which indicates obvious shortcomings of the interpolation algorithm when processing data sets containing points located at a relatively small distance from each other.

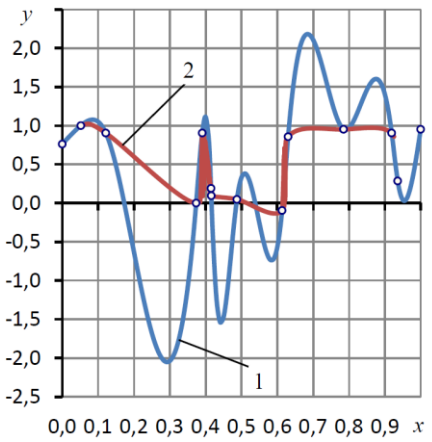

Figure 3. Curves of the B-spline (1) and the Catmull-Rom spline (2)

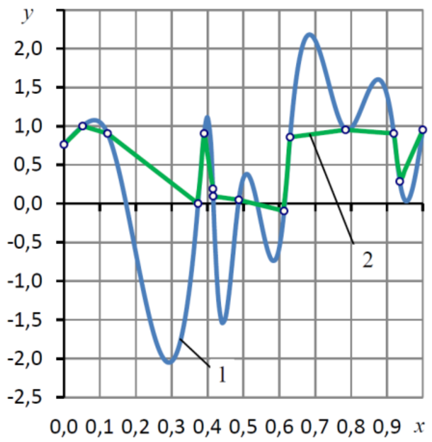

Figure 4. B-spline (1) and linear interpolation spline (2)

Below, we consider a technique for constructing a spline, which is to a certain extent a compromise with respect to the mentioned splines: it is characterized by smaller «ejections» of *B*-splines and significantly less pronounced kinks of Akima splines.

2. Construction of third degree directional spline (DS3-spline)

On each segment $[x_{i-1}, x_i]$ we will represent the spline $S(x)$ as a third degree polynomial

$$S_i(x) = a_i + t\left(b_i + t\left(c_i + d_i t\right)\right), t = \left(x - x_i\right)/h_i, i = 1, 2, ..., n. \qquad (3)$$

At the junctions of adjacent segments $[x_{i-1}, x_i]$ for polynomials (3), it is required to fulfill the continuity conditions for spline and its first derivative at points (1)

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$$S_i(x_{i-1}) = S_{i-1}(x_{i-1}), \quad S_i'(x_{i-1}) = S_{i-1}'(x_{i-1}). \qquad (4), (5)$$

Using (3) – (5) we get

$$\begin{cases} a_i = y_i, \\ a_i - b_i + c_i - d_i = y_{i-1}, \\ b_i - 2c_i + 3d_i = b_{i-1}. \end{cases} \qquad (6)$$

If we consider b_i known, then (6) allows us to obtain a system of equations for unknown coefficients c_i, d_i

$$\begin{cases} c_i - d_i = b_i - v_i, \\ 2c_i - 3d_i = b_i - b_{i-1}, \end{cases} \qquad (7)$$

where

$$v_i = y_i - y_{i-1}.$$

Having solved (3) we find

$$c_i = 2b_i + b_{i-1} - 3v_i, \quad d_i = b_i + b_{i-1} - 2v_i. \qquad (8)$$

In the simplest case, for the extreme segments $[x_0, x_1]$ and $[x_{n-1}, x_n]$, we can use the derivative of the linear interpolation spline by setting

$$b_0 = v_1, \quad b_n = v_n.$$

The same coefficients can be determined using the three-point construction (x_{i-1}, y_{i-1}), (x_i, y_i), (x_{i+1}, y_{i+1}) of the Lagrange interpolation polynomial [Atkinson, 1988]

$$f_i(x) = y_i + \frac{\left[u_i h_{i+1} + u_{i+1} h_i + (u_{i+1} - u_i)(x - x_i)\right](x - x_i)}{h_{i+1} + h_i}, \quad i = 1, 2, ..., n-1,$$

$$b_0 = f_1'(x_0) = u_1 - \frac{h_1(u_2 - u_1)}{h_1 + h_2}, \quad b_n = f_{n-1}'(x_n) = u_n + \frac{h_n(u_n - u_{n-1})}{h_{n-1} + h_n}.$$

For intermediate segments $[x_{i-1}, x_i]$ it is used the formula

$$b_i = \alpha u_i + (1 - \alpha) u_{i+1}, \qquad (9)$$

where coefficient $\alpha \in [0, 1]$.

From (4) – (9) it follows that on the segment $[x_{i-1}, x_i]$ the function $S_i(x)$ is completely determined by only three points, while the locality of the Akima spline is determined by five points [Akima, 1970], and the *B*-spline by all points of the set (1). This property characterizes the best comparative speed of the method in the correction of coefficients (3) in case of changes in individual points of the set (1).

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The divided difference u_i is the angular coefficient of the corresponding segment of the linear interpolation spline, which is shown in Figure 4.

Obviously, (9) is the angular coefficient of the tangent at the point (x_i, y_i) of the conjugation of the spline segments located on both sides of it. By varying the coefficient α, one can adjust the position of the tangents at the intermediate points of the spline, directing it so that the «ejections» are minimal and the kinks are not too noticeable. We call such spline as *directional*, and α as *directional coefficient*.

In the simplest case, we can set α = 0.5, that is, assume that the guide tangent to the spline at intermediate nodal points should occupy a middle position relative to adjacent segments of the linear interpolation spline.

Figure 5 shows an example of interpolating a data set using a *B*-spline (1), Akima spline (2) and directional spline (3).

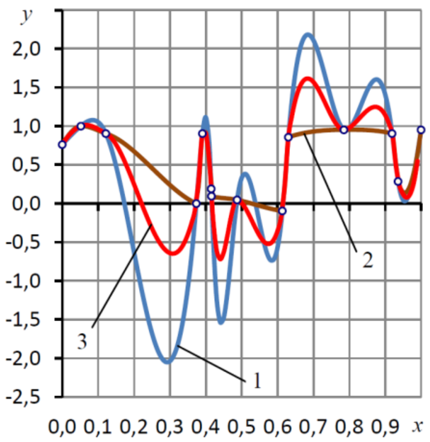

Figure 5. *B*-spline (1), Akima spline (2) and directional spline (3)

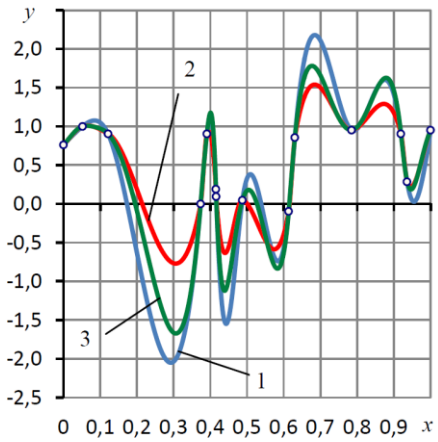

Figure 6. B-spline (1), DS3-spline (2), DS4-spline (3)

It can be seen that, unlike the Akima spline, the directional spline does not have visible kinks, which for many practical applications turns out to be a sufficient basis for deciding on a satisfactory approximation of the original table function by such splines. Compared to the *B*-spline, spline «ejections» are less pronounced.

When performing a computational experiment on thousands of random sets (1), a directional spline invariably showed resistance to «ejections» and kinks.

3. Optimization of DS3-spline

The aim of optimization is to make the kinks of the directional spline less noticeable, which are determined by the absolute value of the difference in the values of the second derivatives at the conjugation point of neighboring

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polynomials. The magnitude of such a gap at intermediate points can be determined by the formula

$$D_i(\alpha) = \frac{\varepsilon}{2}\left|S_i''(x_{i-1}) - S_{i-1}''(x_{i-1})\right| = \left|c_i - c_{i-1} - 3d_i\right|, (i = 1, 2, ..., n-1). \quad (10)$$

A computational experiment showed that in most cases, at extreme values of the directing coefficient α = 0 and α = 1, when the angles of the tangent to the curve at the intermediate points and one of the segments of the linear spline coincide, the directional spline usually has not only pronounced kinks, but also large «ejections». At intermediate values of α, these disadvantages are less noticeable. Therefore, there are α values at which these shortcomings will be least pronounced.

The essence of optimization is to find such $\alpha = \alpha_{opt}$ at which the largest gap

$$D(\alpha) = \mathrm{Max}\, D_i(\alpha), \ (i = 1, 2, \ldots, n-1) \quad (11)$$

will be minimal.

It is established that in most cases $D(\alpha)$ is piecewise linear functions with a single break point. However, there are frequent cases of functions with several kinks.

Cases are noted where the function may not have an extremum, that is, its minimum is at one of the edges of the interval of variation of α. In such cases, it is advisable to take α = 0.5.

The value of α_{opt} can be found using one of the methods for minimizing unimodal functions [Kiefer, 1953; Brent, 1973; Dobson, 1983]. However, one can use the piecewise linearity property $D(\alpha)$ and, on this basis, propose a faster algorithm.

Problem solving method and a description of the algorithm for finding α_{opt} are given below. Values of the type $T = (T.x, T.y, T.z)$, are used, where $T.x$, $T.y$ are the abscissa and the ordinate of the point, $T.z$ is the value of the derivative of the function D (α) at this point.

Step 1. Set a sufficiently small number ε – determination accuracy α_{opt}, as well as limit range $A.x = 0$; $B.x = 1$.

Step 2. Find the value of the function $A.y = D(A.x)$ and $A.z = [D(A.x+\varepsilon) - A.y]/\varepsilon$ at the left end of the search interval. We define $B.y = D(B.x)$ and $B.z = [B.y - D(B.x-\varepsilon)]/\varepsilon$ – their analogous values at the right end of the search interval.

Step 3. Using these points and derivatives we construct straight lines, the first of which passes through the point $(A.x, A.y)$, the second through $(B.x, B.y)$. It is easy to show that the abscissa of the intersection point of these lines can be found by the formula

$$x = \frac{B.y - A.y - B.zB.x + A.zA.x}{A.z - B.z}. \quad (12)$$

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We calculate the *abscissa* $C.x = x+\varepsilon/3$ shifted to the right of x by an amount less than ε. This is necessary so that point x falls into the range [$A.x$, $C.x$].

Step 4. If $|C.x–A.x| <\varepsilon$ or $|C.x–B.x| <\varepsilon$, then the minimum point $\alpha_{opt} = x$ is found and the algorithm finishes its work; otherwise, we find similar values of $C.y$ and $C.z$. If $C.z$ and $A.z$ are numbers of the same sign, then we set $B = C$, otherwise $A = C$ and go to step 2 to perform a new iteration.

Let us demonstrate the operation of the algorithm by the example of optimization of the function, which is shown in Figure. 6.

To do this, in step 1, we set the search accuracy $\varepsilon = 10^{-3}$.

In step 2, we obtain $A.x = 0$; $A.y = 1.2$; $A.z = –1{,}7$; $B.x = 0.999$; $B.y = 0.6$; $B.z = 1.7$. Different signs of the derivatives $A.z$ and $B.z$ indicate that the function $D(\alpha)$ is unimodal [Kodnyanko – 1, 2019] and, therefore, its minimum is inside the segment.

At step 3, by the formula (12) we find $x = 0.644$; $C.x = 0.645$. Since at step 4 none of its conditions were met, we calculate $C.y$ and $C.z = 0{,}8 > 0$. This means that the function increases to the right of x, therefore, the minimum is to the left of x. We set $B = C$ and continue the search for the minimum of the function on the interval [0; 0. 645] of shorter length.

At the new iteration, we get $x = 0.425$; $C.x = 0.426$; $C.z = 0.8 > 0$ and a new segment [0; 426].

At the next iteration, we find $x = 0.425$; $C.x = 0.426$. Now the condition $|C.x–B.x| <\varepsilon$ is fulfilled. This means that the minimum of the function is at the point $\alpha_{opt} = x = 0.425$.

Obviously, the number of iterations does not exceed k+1, where k is the number of kinks of the function $D(\alpha)$. In particular, this problem was solved in three iterations with two kinks of the minimized function.

Table 1. Comparative characteristics of errors when calculating a function using splines

x	$y(x)$	Δ_1	Δ_2	Δ_3
0,00	0,0000000	0,000E+00	0,000E+00	0,000E+00
0,05	0,0499792	7,819E-08	-6,100E-05	3,529E-05
0,10	0,0998334	3,189E-08	-1,010E-05	-5,012E-06
0,15	0,1494381	1,433E-08	7,158E-06	9,359E-06
0,20	0,1986693	8,864E-09	2,348E-05	-5,406E-06
0,25	0,2474040	2,986E-08	-1,177E-05	3,179E-06
0,30	0,2955202	8,974E-11	-1,624E-06	6,155E-07
0,35	0,3428978	2,180E-08	6,243E-06	-3,199E-06
0,40	0,3894183	2,905E-08	-8,113E-06	4,885E-06
0,45	0,4349655	-2,711E-20	-2,711E-20	-5,421E-20
0,50	0,4794255	4,830E-10	5,141E-06	-4,438E-06
0,55	0,5226872	8,824E-08	-2,120E-06	2,857E-06

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0,60	0,5646425	-6,664E-08	-6,810E-06	2,393E-06
0,65	0,6051864	-1,044E-07	4,300E-06	-1,165E-06
0,70	0,6442177	8,651E-07	3,549E-06	-2,821E-06
0,75	0,6816388	-2,716E-06	-1,707E-06	1,296E-06

In conclusion, we present a table of values of the function $y(x) = \sin x$ used in [Moreau, 1981] to estimate the error of the *B*-spline and Akima spline. Splines are built on a set of 14 random points. The calculations are carried out for equally spaced nodes. For comparison, directional spline data has been added to the table. Table 1 in columns Δ_1, Δ_2, Δ_2 shows the differences between the exact value of the function $y(x)$ and the values obtained using the *B*-spline (Δ_1), Akima spline (Δ_2) and directional spline (Δ_3), respectively.

B-spline showed higher accuracy. Among the last two, directional spline gave the best results. This is not an obvious result, since it was expected that the function $y(x)$ supposedly has advantages in interpolating monotone functions, namely, in this example, the Akima spline should have shown better performance not only in comparison with the directional spline, but also with *B*-spline. However, in this case, these expectations were not realised.

The presented idea allows to expand the scope of the approach to constructing a directional new spline that will be free from kinks and allows the possibility of its optimization to suppress emissions. The example is a directional spline of the fourth degree, the construction method of which is described below.

4. Construction of fourth degree directional spline (DS4-spline)

On each segment $[x_{i-1}, x_i]$ we will represent the spline $S(x)$ as a polynomial of the fourth degree

$$S_i(x) = a_i + b_i\left(x - x_i\right) + c_i\left(x - x_i\right)^2 + d_i\left(x - x_{i-1}\right)^3 + e_i\left(x - x_{i-1}\right)^4, i = 1, 2, ..., n. \quad (13)$$

At the joints of adjacent segments $[x_{i-1}, x_i]$ for polynomials (3), we require the conditions for the spline and its first and second derivatives at the points (1) be satisfied

$$S_i(x_{i-1}) = S_{i-1}(x_{i-1}), \quad S_i'(x_{i-1}) = S_{i-1}'(x_{i-1}), \quad S_i''(x_{i-1}) = S_{i-1}''(x_{i-1}). \quad (14), (15), (16)$$

Assuming, as before, b_i are known and using (14), we obtain

$$c_i = d_i h_i - e_i h_i^2 + w_i, \quad e_i = \frac{w_i - c_i + d_i h_i}{h_i^2}, \quad (17), (18)$$

where

$$w_i = \frac{b_i - u_i}{h_i}.$$

Using (15) we find

DOI: 10.24818/18423264/54.3.20.08

$$2c_i + 4e_i h_i^2 = v_i - \frac{3h_{i-1}^2}{h_i} d_{i-1}, \tag{19}$$

where

$$v_i = \frac{b_i - b_{i-1}}{h_i}.$$

Substituting (18) into (19) we write

$$c_i = 2d_i h_i + \frac{3h_{i-1}^2}{2h_i} d_{i-1} + 2w_i - \frac{v_i}{2}, \quad e_i h_i^2 = \frac{v_i}{2} - d_i h_i - \frac{3h_{i-1}^2}{2h_i} d_{i-1} - w_i. \qquad (20), (21)$$

Condition (16) gives the dependence

$$c_i + 6e_i h_i^2 = c_{i-1} + 3d_{i-1} h_i. \tag{22}$$

Substituting (20), (21) into (22) we find

$$8d_i h_i + d_{i-1}\left(15\frac{h_{i-1}^2}{h_i} + 10h_{i-1}\right) + \frac{3h_{i-2}^2}{h_{i-1}} d_{i-2} = 5v_i + v_{i-1} - 4\left(2w_i + w_{i-1}\right). \tag{23}$$

Shifting the index in (23), we obtain recurrence formulas

$$\mu_i d_{i+1} + \eta_i d_i + \lambda_i d_{i-1} = \omega_i, \quad i = 1, 2, ..., n-1, \tag{24}$$

where

$$\mu_i = 8h_{i+1}, \eta_i = 5h_i\left(3\frac{h_i^2}{h_{i+1}} + 2\right), \lambda_i = \frac{3h_{i-1}^2}{h_i}, \omega_i = 5v_{i+1} + v_i - 4\left(2w_{i+1} + w_i\right).$$

Formula (24) is a three-diagonal system of linear algebraic equations for unknown coefficients d_i, which, taking into account the obvious boundary conditions $d_0 = 0, d_n = 0$, can be solved by the sweep method [Catmull and Rom, 1974]. Further, the coefficients c_i, e_i can be found by formulas (17), (18).

5. Optimization of DS4-spline

Optimizing the DS3-spline, a one-parameter procedure was used. This is due to the fact that multi-parameter optimization in the limit gives Schoenberg *B*-spline, which results in the loss of DS3-spline advantages. The DS4-spline is deprived of kinks; thus, its optimization is reduced to only maximum suppression of "outliers". In this process, all guiding spline coefficients α_i can be involved, with the help of which the DS4-spline coefficients are calculated

$$b_i = \alpha_i u_i + (1 - \alpha_i) u_{i+1}, \quad i = 1, 2, ..., n-1.$$

As criteria for optimality of this spline were used
- the length *L* of the spline,
- the difference *R* between its global maximum and global minimum.

Criterion *L* is determined by the sum of the lengths of the spline segments and can be calculated using the well-known formula [Dietze and Schmidt, 1988]

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$$L = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \sqrt{1+\left[s_i'(x)\right]^2}\,dx, \qquad (25)$$

where

$$s_i'(x) = b_i + 2c_i\left(x - x_i\right) + 3d_i\left(x - x_{i-1}\right)^2 + 4e_i\left(x - x_i\right)^3. \qquad (26)$$

To calculate the criterion *R*, formula (26) was used, as well as the formula

$$s_i''(x) = 2c_i + 6d_i\left(x - x_{i-1}\right) + 12e_i\left(x - x_i\right)^2. \qquad (27)$$

Using (26) we found the zeros of the equation and their type was controlled by using (27).

Thus, both criterion *L* and criterion *R* are functions of many variables

$$K = K\left(\alpha\right) \qquad (28)$$

where

$$\alpha = \left(\alpha_1, \alpha_2, ..., \alpha_{n-1}\right).$$

Obviously, Schoenberg's *B*-spline is a special case of the DS4-spline. Therefore, from the position of a minimum of these criteria, the optimal spline cannot be worse than the Schoenberg spline.

In the process of minimizing the criteria, the requirements were followed for maintaining such shape of splines, the iso-geometry of which would correspond to the Schoenberg spline [Miroshnichenko, 1995].

Calculations of the DS4-spline showed that without taking measures, the spline can lose the specified shape.

Among the reasons for the loss of shape are the following:

– the spline may have protrusions of individual fragments of the curve,

– new local extremes may appear on the curve,

– new points for changing the spline curvature may also appear.

These shortcomings are associated with the emergence of new local extremes, both the spline itself and its first and second derivatives, as well as new sign changes of its third derivative.

During optimization, variants of such splines were rejected.

Numerical experiments have made it possible to establish the fact that function (28) is multi-extreme, that is, it has many local minima, among which one should find a global minimum that corresponds to the spline of the optimal shape.

So, for n = 12, for which most experiments were carried out, one would have to find the global minimum of the function of 11 variables, which seems to be an extremely difficult task. In the general case, the difficulties in obtaining a solution to such problem cast doubt on the value of the practical use of the spline under discussion.

The way out was found by using the mentioned properties of the spline, which are dictated by the stringent conditions for maintaining its shape.

The technique of minimizing the criteria is as follows.

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Initially, a starting state is established for which a vector α is selected, all components of which are assumed to be equal to 0.5, and a starting DS4- spline is calculated.

Next, a small-length step τ is assigned and one-parameter minimization is performed for each component of the vector α, for which the process starts from the starting state. The best coordinate wise splines obtained in this way give a lot of α – vectors, the number of which is $m< n$. Observations showed that no more than half of the initial starts usually pass through the filter of stringent requirements to preserve the shape of the spline. So, for example, for $n = 12$, usually $m < 7$.

The next step is a similar one-parameter optimization, where vectors that have passed the filter of the first step are sequentially used as starts. Moreover, the number of new vectors that have passed the shape-preservation filter is also small and it is usually less than the same amount as the previous step.

Recursion is carried out until none of the filtered vectors gives new vectors to continue the process. The result of optimization will be the spline with the lowest value of the criterion K. The calculations showed that, for example, for $n = 12$ it is usually required to form a spline and calculate its characteristics 1000-2000 times.

Figure 6 shows the curves of the splines, which give a typical picture of the shape of the DS4-spline that is optimal according to the criterion.

Such spline usually occupies a middle position between the Schoenberg spline and DS3-spline is closer to the first.

As follows from the above data, the Schoenberg spline length is 19.70. The initial DS4-spline has a slightly longer length - 19.90. The optimized spline has a length of 16.82, which is 14.6% less than the Schoenberg spline. During the optimization, the algorithm improved the result 181 times. To solve the problem, it was necessary to calculate the spline 1312 times.

The smallest length is DS3-spline. With a length of 11.58, it is almost two times shorter than the Schoenberg spline.

Thousands of computational experiments performed for $n = 12$ showed that the optimized DS4-spline is shorter by 5–50% than the Schoenberg spline, and DS3-spline by 1.5–3 times shorter.

Examples of optimization processes in dynamics can be observed on videos at the hyperlinks [Kodnyanko – 2020a, 2020b,2020c].

Experiments have shown that DS4-spline can significantly attenuate the manifestation of "emissions" Schoenberg spline.

However, the best in this respect should be recognized DS3-spline, on which there are no emissions, and kinks are manifested very weakly, in many cases in practical implementations they are insignificant.

In the process of studying the properties of the proposed splines, it was found that in addition to “outliers” and kinks, splines can have sharp peaks of local maxima and minima, which in some cases should be considered as interpolation

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shortcomings. Below is proposed and described a method that allows blunting the "emissions" and peaks of extrema.

6. Technique for blunting sharp spline vertices

Let it be necessary to blunt the local minimum of any of the considered splines and let (x_c, y_c) be the point of such a minimum. We also consider the inflection points (x_a, y_a) and (x_b, y_b) located on both sides of the extremum.

We introduce the local function of the spline

$$g(x) = G\left(x - x_a\right)^k \left(x_b - x\right)^m, G \geq 0, k > 2, m > 2. \qquad (29)$$

For its, there are obvious boundary conditions

$$g'(x_a) = g'(x_b) = 0, g''(x_a) = g''(x_b) = 0.$$

Therefore, at the edges of the interval, the function does not introduce kinks and contributes to blunting of the spline in the region of its minimum.

The maximum of this function takes place at

$$g'(x_c) = G\left(x_c - x_a\right)^{k-1} \left(x_b - x_c\right)^{m-1} \left[k\left(x_b - x_c\right) - m\left(x_c - x_a\right)\right] = 0 \qquad (30)$$

so

$$m = tk, \quad t = \frac{\left(x_b - x_c\right)}{\left(x_c - x_a\right)}.$$

If $t < 1$, then we take $m = 2 + \varepsilon$, where ε is a small number. Then $k = m/t > 2$. Otherwise, for $t \geq 1$, we take $k = 2 + \varepsilon$. Then $m = kt > 2$.

Turn to linear interpolation spline

$$L(x) = L_a x + L_b, L(x_a) = y_a, L(x_b) = y_b,$$

where

$$L_a = \frac{y_b - y_a}{x_b - x_a}, L_a = \frac{x_b y_a - x_a y_b}{x_b - x_a}.$$

For the point of the considered spline minimum, we have

$$G\left(x_c - x_a\right)^k \left(x_b - x_c\right)^m = \sigma\left[L(x_c) - y_c\right], \qquad (31)$$

where σ is the peak bluntness coefficient.

For σ = 0, the addition of g (x) is absent; for σ = 1, the total spline $q(x) = s(x)+g(x)$ touches the line segment $L(x)$, which is the limiting case, since for σ> 1 the total the spline will receive a new extreme maximum, on both sides of which two new minimums will be formed. Such modes do not ensure the preservation of the spline shape, therefore $0 \leq \sigma \leq 1$.

To maintain the shape, the total spline must also have a curvature of the same sign up to $\sigma = \sigma_{max} < 1$. A given value can be determined from the condition

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$$q''(x_c) = 0.$$

Having differentiated the function q (x), we find

$$s''(x_c) + G(k+m)G\left(x_c - x_a\right)^{k-1}\left(x_b - x_c\right)^{m-1} = 0.$$

It follows

$$G = \frac{s''(x_c)}{(k+m)\left(x_c - x_a\right)^{k-1}\left(x_b - x_c\right)^{m-1}}. \quad (32)$$

From (31), (32) it turned out that

$$\frac{\sigma_{\max}\left[L(x_c) - y_c\right]}{\left(x_c - x_a\right)^{k-1}\left(x_b - x_c\right)^{m-1}} = \frac{s''(x_c)}{(k+m)\left(x_c - x_a\right)^{k-1}\left(x_b - x_c\right)^{m-1}}. \quad (33)$$

Relation (33) allows us to determine the maximum permissible coefficient σ

$$\sigma_{\max} = \frac{s''(x_c)\left(x_c - x_a\right)\left(x_b - x_c\right)}{(k+m)\left[L(x_c) - y_c\right]}.$$

The bluntness of local maxima is carried out similarly.

The decision on the extremes to be blunted can be made both in automated and automatic modes.

Making automatic decision, a quantity can be used as a criterion for evaluating the extremum $k_e = s''(x_c)$.

Here is a description of the methodology for assessing extrema, subjected to automatic blunting.

We will use DS3-spline, the graph of which is given in Figure 5. In visual assessment of the spline, two extrema can be distinguished, which must be dulled. One of them is a maximum with an abscissa x = 0.395, the second is a neighboring minimum with an abscissa x = 0.440. For the first k_e = –2920, for the second k_e = 1871. The k_e right criterion that follows in decreasing absolute value is the extreme right minimum with an abscissa x = 0.946, for which k_e = 1123. Let us evaluate it as an extremum that does not require blunting.

This simple analysis allows, as a first approximation, to formulate the following automatic expert assessment methodology: extrema that should be blunted automatically must satisfy the condition | k_e | > 1200, if this does not contradict conditions for maintaining the shape of the spline.

In Figure 7, as an example of blunting extrema, a linear interpolation spline, a *B*-spline, a DS3-spline and two fragments of a DS3-spline are shown, on which the extrema are blunted automatically in accordance with the method described above. In the calculation, the bluntness coefficient σ = 0,75 σ_{max} was adopted.

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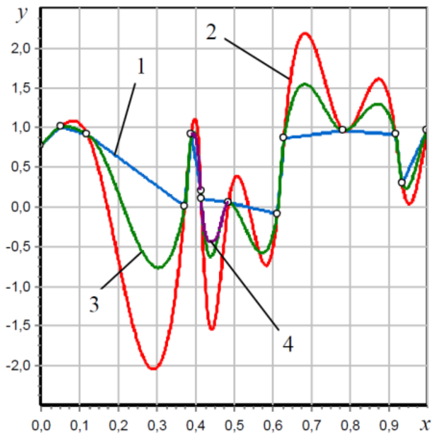

Figure 7. Linear interpolation spline (1), *B*-spline (2), DS3-spline (3) and DS3-spline with blunted extremes (4)

The blunting technique can be applied to splines of any type.

7. Conclusion

The article proposes a method for constructing a cubic spline for a set of points on a plane. The spline is compared with Schoenberg *B*-spline and Akima and Catmull-Rom splines. It is shown that for unequally spaced points, at which the disadvantages of the named splines usually appear, in comparison with the *B*-spline, it gives significantly lower oscillations. A spline with such a set of points is practically deprived of the strong kinks that are characteristic of Akima splines.
It does not give loops and oscillations, which are a characteristic disadvantage of parametric splines, in particular, Hermitian splines, which include the Catmull-Rom splines. A method for optimizing the weight coefficient of the directional spline is proposed, the purpose of which is to minimize the discontinuities of the second derivative of the function at its intermediate points.

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